Network Coding Capacity of Random Wireless Networks under a Signal-to-Interference-and-Noise Model

Zhenning Kong*, Salah A. Aly†, Emina Soljanin‡, Edmund M. Yeh*, and Andreas Klappenecker†
*Department of Electrical Engineering, Yale University, New Haven, CT 06520, USA
Email: {zhenning.kong, edmund.yeh}@yale.edu
†Department of Computer Science, Texas A&M University, College Station, TX 77843, USA
Email: {salah, klappi}@cs.tamu.edu
‡Bell Laboratories, Alcatel-Lucent, Murray Hill, NJ 07974, USA
Email: emina@lucent.com

***Abstract*— In this paper, we study network coding capacity for random wireless networks. Previous work on network coding capacity for wired and wireless networks have focused on the case where the capacities of links in the network are independent. In this paper, we consider a more realistic model, where wireless networks are modeled by random geometric graphs with interference and noise. In this model, the capacities of links are not independent. We consider two scenarios, single source multiple destinations and multiple sources multiple destinations. In the first scenario, employing coupling and martingale methods, we show that the network coding capacity for random wireless networks still exhibits a concentration behavior around the mean value of the minimum cut under some mild conditions. Furthermore, we establish upper and lower bounds on the network coding capacity for dependent and independent nodes. In the second one, we also show that the network coding capacity still follows a concentration behavior. Our simulation results confirm our theoretical predictions.**

I. Introduction

Network coding was originally proposed by Ahlswede *et al.* in [1]. Unlike traditional store-and-forward routing algorithms, in network coding schemes, intermediate nodes encode their received messages and forward the coded messages to their next-hop neighbors. It has been shown that network coding can improve the network capacity, even by using simple linear or random codes [8], [9], [11], [12]. In most studies of network coding, network topologies are assumed to be known.

In [17], [18], the authors studied network coding capacity for weighted random graphs and random geometric graphs. In the random graph model, each pair of nodes are connected by a bidirectional link with probability $p < 1$ independently [4], [10]. The capacity of each link is assumed to be i.i.d. according to some probability distribution. In the random geometric graph model, two nodes are connected to each other by a bidirectional link only when their distance is less than a predefined positive value r, the characteristic radius [15]. Each link has a unit capacity. For these two types of random networks, the authors showed that the network coding capacity is concentrated at the (weighted) mean degree of the graph, i.e., the (weighted) mean number of neighbors of each node. Essentially, the results reveal a concentration behavior of the size of the minimum cut between two nodes in random graphs or random geometric graphs. Similar problems have been studied in the literature, e.g., [6] and references there. In [3], the authors studied a generalized random geometric graph model, where two nodes are connected by a bidirectional link with probability 1 if their distance d is less than $r_0 > 0$ and with probability $p < 1$ if $r_0 < d \leq r_1$. They obtained similar concentration results there.

The geometric models in [3], [17], [18] assume that a link exists (possibly with a probability) between two nodes when the nodes are within each other's transmission range. Although each link has a direction, as all links are bidirectional (i.e., the link (i, j) implies the existence of the link (j, i)), the model in fact leads to an *undirected* graph and considerably simplifies the resulting analysis. In addition, interferences among wireless terminals were not considered in [3], [17], [18]. Nevertheless, in wireless networks, due to noise, interference, and heterogeneity of transmission power, significantly more sophisticated models for link connectivity are needed. For instance, a widely-used model for wireless communication channels is the Signal-to-Interference-plus-Noise-Ratio (SINR) model [16], [19]. In this paper, we study the capacity, i.e., the size of the minimum cut, of random wireless networks under the SINR model.

Since how to apply the network coding with noisy links is still an open problem, we assume that as long as the SINR of a link (i, j), β_{ij} is greater than or equal to a predefined threshold β, then node i can transmit data at rate R packets/sec to node j without any error. That is links are noise-free once the SINR condition is met. In other words, we view the network coding as operation on a higher layer in the network communication stack, and assume there is an error correcting code at the lower layer which corrects errors on the links once the SINR threshold is met. Then, in this model, each link is indeed directional (not necessarily bidirectional), and the capacities of different links are not independent. We will show that the capacity still has a sharp concentration when the scale of the network is large enough.

This paper is organized as follows. In Section II, we describe the random wireless network model. In Section III, we study the network coding capacity for a single source and multiple destinations transmissions. Specifically, we investigate two cases. In the first one, all nodes have

the same transmission power, and in the second one, the transmission powers are heterogeneous. We use different techniques for these two cases and show that the network coding capacity has a concentration behavior in both cases. In Section IV, we extend our result to multiple sources and multiple destinations transmission problem. In Section V, we present some simulation results, and finally, we conclude this paper in Section VI.

II. Random Wireless Networks Model

We use the following model for random wireless networks. Assume

(i) $\mathcal{X} = \{\mathbf{X}_1, \mathbf{X}_2, ..., \mathbf{X}_n\}$ is a set of i.i.d. two-dimensional random variables according to a homogeneous Poisson point process in the two-dimensional unit torus, where $\mathbf{X}_i$ denotes the random location of node i, and n is the total number of nodes.

(ii) Each node i has a transmission power P_i, which follows a probability distribution $f_P(p)$, $p \in [p_{min}, p_{max}]$, where $0 < p_{min} \leq p_{max} < \infty$.

Here, the existence of a link from node i to node j depends on the ability to decode the transmitted signal from i to j, which is determined by the Signal-to-interference-plus-noise-ratio (SINR) given by

$$\beta_{ij} = \frac{P_i L(d_{ij})}{N_0 + \gamma \sum_{k \neq i,j} P_k L(d_{kj})}, \tag{1}$$

where P_i is the transmission power of node i, d_{ij} is the distance between nodes i and j, and N_0 is the power of background noise. The parameter γ is the inverse of system processing gain. It is equal to 1 in a narrow-band system and smaller than 1 in a broadband (e.g., CDMA) system. The signal attenuation function $L(\cdot)$ is a function of the distance $d_{ij} = ||\mathbf{X}_i - \mathbf{X}_j||$, where $\|\cdot\|$ is the Euclidean norm, and is usually given by $L(d_{ij}) = c d_{ij}^{-\alpha}$ for some constants c and $2 < \alpha < 4$.

Under the SINR model, the transmitted signal of node i can be decoded at j if and only if $\beta_{ij} > \beta$, where β is some threshold for decoding. In this case, a link (i,j) is said to exist from i to j. Note that even if $\beta_{ij} > \beta$, $\beta_{ji} > \beta$ may not hold and thus the link (j,i) may not exist. Thus, the graph resulting from the SINR model is in general *directed*. It is clear that link (i,j) is bidirectional if and only if $\min\{\beta_{ij}, \beta_{ji}\} > \beta$. Denote by $G(\mathcal{X}, \mathcal{P}, \gamma)$ the ensemble of random wireless networks induced by the above physical model, where $\mathcal{P} = \{P_1, P_2, ..., P_n\}$ represents the set of transmission power.

For transmission power P and signal attenuation function $L(\cdot)$, we assume

(i) $p_{min} > \beta N_0$;
(ii) $\Pr(P = p_{min}) > 0, \Pr(P = p_{max}) > 0$,
(iii) $L(x)$ is continuous and strictly decreasing in x

for technical and practical reasons. In the remainder of this paper, under different circumstances, we may add further constrains on P_i.

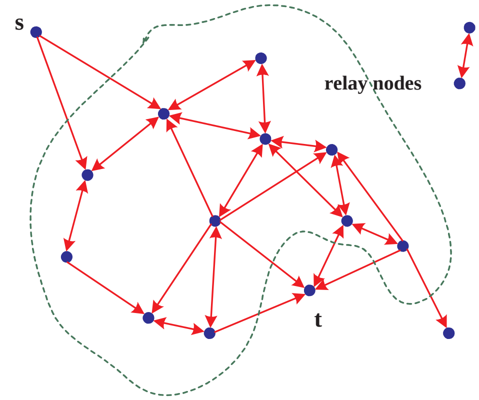

Fig. 1. Single-source single-destination transmission in directed SINR graphs

The sum $\sum_{k \neq j} L(d_{kj}) = \sum_{k \neq j} L(||\mathbf{X}_k - \mathbf{X}_j||)$ is a random variable depending on the locations of all nodes in the network. Define

$$J(j) \triangleq \sum_{k \neq j} L(d_{kj}), \quad \text{for all } j, \tag{2}$$

$$I(j) \triangleq \sum_{k \neq j} P_k L(d_{kj}), \quad \text{for all } j. \tag{3}$$

To study the asymptotic network capacity, we will let the number of nodes n go to infinity. Since the region is fixed, this corresponds to a dense network model [**?**], [15]. Another widely used model is the extended network model [7], [13], in which the number of nodes and the area of the region both go to infinity while the ratio between them—the density of the network, is kept as a constant. Both models are widely used in the literature. We will focus on the former one in this paper.

III. Network Coding Capacity for Single Source Transmission

A. Capacity of a Cut

Let C_{ij} be the capacity of a link (i,j). We will specify the value of C_{ij} later for different scenarios. Consider a single-source multiple-destination transmission problem. Let s be the source node. Suppose there are l destination nodes, $t_1, ...t_l$, and m relay nodes, $u_1, ...u_m$. Denote the set of the destination nodes and relay nodes by $\mathcal{T}$ and $\mathcal{R}$, respectively. Fig. 1 illustrates an example of single-source single-destination transmission.

Let the capacity of the link from the source s to each relay node u_i be C_{si}, $i = 1, ..., m$, the capacity from relay node u_i to another relay node u_j be C_{ij}, $i \neq j, i = 1, ..., m, j = 1, ..., m$, and the capacity from each relay node u_i to each destination node t_j be C_{it_j}, $i = 1, ..., m, j = 1, ..., l$. Unlike random geometric graph models studied in [3], [17], [18], the capacities in our model are not symmetric nor independent in general.

Since in our random SINR wireless network model, there are two sources of randomness: one is the random location of each node and the other is the random transmission power

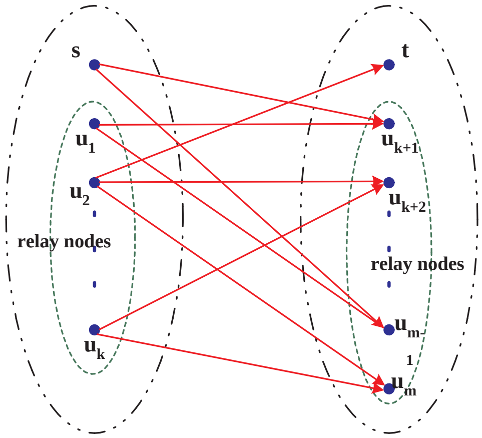

Fig. 2. An s-t-cut for the single-source single-destination transmission in directed SINR graphs

of each node. We use E_X and E_P to denote the expectation operation with respect to each probability measure respectively.

Let $\bar{C}$ be the expected capacity of a link (i,j) which is defined as

$$\begin{aligned}\bar{C} &= E_X E_P[C_{ij}] \\ &= \int_0^{\frac{p_{max}}{N_0}} C_{ij} dF_{\beta_{ij}}(\tau),\end{aligned} \tag{4}$$

where $F_{\beta_{ij}}(\cdot)$ is the c.d.f. of β_{ij}, which is determined by $f_P(\cdot)$, the distribution of $\mathcal{X}$, and path-loss function $L(\cdot)$.

Now define an s-t-cut of size k for a pair of given source s and destination $t \in \mathcal{T}$ as a partition of the relay nodes into two sets V_k and V_k^c, such that $|V_k| = k, |V_k^c| = m-k$, $V_k \cup V_k^c = \mathcal{R}$ and $V_k \cap V_k^c = \emptyset$. An example of an s-t-cut is shown in Fig. 2. Let

$$C_k = \sum_{u_i \in V_k^c} C_{si} + \sum_{u_j \in V_k} \sum_{u_i \in V_k^c} C_{ji} + \sum_{u_j \in V_k} C_{jt}, \tag{5}$$

then C_k is the capacity of the corresponding s-t-cut. Although C_k is a sum of dependent random variables, we still have

$$\begin{aligned}E[C_k] &= E_X E_P[C_k] \\ &= \sum_{u_i \in V_k^c} E_X E_P[C_{si}] + \sum_{u_j \in V_k} \sum_{u_i \in V_k^c} E_X E_P[C_{ji}] \\ &\quad + \sum_{u_j \in V_k} E_X E_P[C_{jt}] \\ &= [m + k(m-k)]\bar{C},\end{aligned} \tag{6}$$

and consequently $E[C_k] = E[C_{m-k}]$ for $k = 0, 1, ..., m$, and $E[C_0] \leq E[C_1] \leq \cdots \leq E[C_{\lceil m/2 \rceil}]$.

To show the capacity of any source-destination pair concentrates at some value, we will first show that for such a source-destination pair, the capacity of any s-t-cut of size k concentrates at its mean value. Similar results were proved in [3], [17], [18], where the capacities of the links that originate from the same node are i.i.d. Nevertheless, the methods used in [3], [17], [18] do not apply here, since in the SINR model, C_k is a sum of dependent random link capacities. Instead, we employ coupling, martingale methods and Azuma's inequality [14], [15] to solve the problem for different cases.

Note that when $\gamma = 0$, i.e., there is no interference in the networks, the capacities C_{si} for $i = k+1, ..., m$ are mutually independent, as well as the capacities C_{ij} for any fixed $i = 1, ..., k$ with $j = k+1, ..., m$ or t. In this case, although the link capacities are still asymmetric, $\sum_{u_i \in V_k^c} C_{si}$ and $\sum_{u_i \in V_k^c \cup \{t\}} C_{ji}$ for $j \in V_k$ become sums of independent random variables. Thus we can apply methods similar to those used in [3], [17], [18] to obtain the same concentration results.

B. Constant Transmission Power

Consider the scenario when all nodes transmit with a constant power P_0 and denote the model by $G(\mathcal{X}, P_0, \gamma)$. The SINR of link (i,j) in this case, β_{ij}, can be rewritten as

$$\begin{aligned}\beta_{ij} &= \frac{L(d_{ij})}{N_0/P_0 + \gamma \sum_{k \neq i,j} L(d_{kj})} \\ &= \frac{L(d_{ij})}{N_0/P_0 + \gamma J(j) - \gamma L(d_{ij})}.\end{aligned} \tag{7}$$

Assume when $\beta_{ij} \geq \beta$, the link (i,j) has a capacity R, i.e., node i can transmit data at rate R packets/sec to node j without any error. Then, we can define C_{ij} as

$$C_{ij} = \begin{cases} R & \beta_{ij} \geq \beta, \\ 0 & \beta_{ij} < \beta. \end{cases} \tag{8}$$

Note that when the wireless channel is Gaussian channel, the capacity of link (i,j) is [5]

$$C_{ij} = \begin{cases} \dfrac{1}{2}\log(1 + \beta_{ij}) & \beta_{ij} \geq \beta, \\ 0 & \beta_{ij} < \beta. \end{cases} \tag{9}$$

Our results in this subsection do not rely on any particular expression of C_{ij}, and thus they hold for C_{ij} defined by (8) as well as for C_{ij} defined by (9). Nevertheless, since we consider the application of network coding, it would be more appropriate to focus on the $R-0$ capacity (8), rather than the capacity of Gaussian channel.

Note that β_{ij} and thus C_{ij} are determined by $L(d_{ij})$ and $J(j)$. Because of the i.i.d. distribution of $\mathbf{X}_i$'s, given $\mathbf{X}_j$, d_{ij}'s are independent for all $i \neq j$. Given node j, let

$$E[L] \triangleq E_{\mathbf{X}_i}[L(d_{ij})], \tag{10}$$

then

$$E[J(j)] = E\left[\sum_{i \neq j} L(d_{ij})\right] = (n-1)E[L] \triangleq E[J]. \tag{11}$$

Since our model is a dense network model and the area of the region is fixed, $E[L] = E[L(d_{ij})]$ is a constant and $E[J] = (n-1)E[L]$ scales with n. For different j's, it is clear that $J(j)$'s are not independent, however, they have the same sharp concentration behavior in large scale wireless networks. This is established in the following lemma.

Lemma 1: Suppose there are n nodes in the network, then

$$\Pr(J(j) \leq (1-\epsilon_1)E[J]) = O\left(\frac{1}{n^2}\right), \qquad (12)$$

and

$$\Pr(J(j) \geq (1+\epsilon_1')E[J]) = O\left(\frac{1}{n^2}\right), \qquad (13)$$

for all $j = 1, 2, ..., n$, where $\epsilon_1 = \sqrt{\frac{4\ln n}{(n-1)E[L]}}$ and $\epsilon_1' = \sqrt{\frac{6\ln n}{(n-1)E[L]}}$.

Proof: Given any node j, because $J(j) = \sum_{i\neq j} L(d_{ij})$, and $L(d_{ij})$ are i.i.d. for all $i \neq j$, by the Chernoff bound [2], [14], we have

$$\begin{aligned}\Pr(J(j) \leq (1-\epsilon_1)E[J]) &\leq \exp\left\{-\frac{E[J]\epsilon_1^2}{2}\right\} \\ &= \exp\left\{-\frac{(n-1)E[L]\epsilon_1^2}{2}\right\} \qquad (14)\end{aligned}$$

and

$$\begin{aligned}\Pr(J(j) \geq (1+\epsilon_1')E[J]) &\leq \exp\left\{-\frac{E[J]\epsilon_1'^2}{3}\right\} \\ &= \exp\left\{-\frac{(n-1)E[L]\epsilon_1'^2}{3}\right\} \qquad (15)\end{aligned}$$

Substituting $\epsilon_1 = \sqrt{\frac{4\ln n}{(n-1)E[L]}}$ and $\epsilon_1' = \sqrt{\frac{6\ln n}{(n-1)E[L]}}$ into (14) and (15), we obtain (12) and (13), respectively. □

Lemma 1 shows that when the network is large, i.e., n is sufficiently large, the interference at each node concentrates at $\gamma(n-1)E[L] = \Theta(1)E[L]$. The reason for this is the uniformly (asymptotically Poisson) random distribution of the nodes.

Now define two other types of SINR models $G'(\mathcal{X}, P_0, \gamma)$ and $G''(\mathcal{X}, P_0, \gamma)$ which are coupled with $G(\mathcal{X}, P_0, \gamma)$ such that they have the same point process $\mathcal{X}$ and constant power P_0. Let the SINR of link (i, j) in $G'(\mathcal{X}, P_0, \gamma)$ and $G''(\mathcal{X}, P_0, \gamma)$ be

$$\beta_{ij}' = \frac{L(d_{ij})}{N_0/P_0 + (1+\epsilon_1')\gamma E[J] - \gamma L(d_{ij})} \qquad (16)$$

and

$$\beta_{ij}'' = \frac{L(d_{ij})}{N_0/P_0 + (1-\epsilon_1)\gamma E[J] - \gamma L(d_{ij})}, \qquad (17)$$

respectively.

Let C_{ij}' and C_{ij}'' be the capacity of link (i, j) in $G'(\mathcal{X}, P_0, \gamma)$ and $G''(\mathcal{X}, P_0, \gamma)$, respectively. Since $\epsilon_1 \to 0$ and $\epsilon_1' \to 0$ as $n \to \infty$, C_{ij}' and C_{ij}'' are asymptotically equal to C_{ij}.

The following lemma establishes a concentration result for C_k with constant transmission power by coupling methods.

Lemma 2: For any $0 < \epsilon < 1$, the capacity of an s-t-cut of size $k, k = 0, 1, ..., m$, satisfies

$$\Pr(C_k \leq (1-\epsilon)E[C_k']) \leq \exp\left\{-\frac{E[C_k']\epsilon^2}{2}\right\}\left(1-O\left(\frac{1}{n}\right)\right), \qquad (18)$$

where $E[C_k'] = [m + k(m-k)]\bar{C}'$ and $\bar{C}'$ is the average link capacity in $G'(\mathcal{X}, P_0, \gamma)$, and

$$\Pr(C_k \geq (1+\epsilon)E[C_k'']) \leq \exp\left\{-\frac{E[C_k'']\epsilon^2}{3}\right\}\left(1-O\left(\frac{1}{n}\right)\right), \qquad (19)$$

where $E[C_k''] = [m + k(m-k)]\bar{C}''$ and $\bar{C}''$ is the average link capacity in $G''(\mathcal{X}, P_0, \gamma)$.

Proof: Since for all j, $\{J(j) \geq (1-\epsilon_1)E[J]\}$ and $\{J(j) \leq (1+\epsilon_1')E[J]\}$ are both increasing events.[1] By the FKG inequality [2], [13], [15], we have

$$\begin{aligned}\Pr\left(\bigcap_{j=1}^{n}\{J(j) \geq (1-\epsilon_1)E[J]\}\right) \geq& \prod_{j=1}^{n}\Pr(J(j) \geq (1-\epsilon_1)E[J]) \\ =&\left(1-O\left(\frac{1}{n^2}\right)\right)^n \\ =&1-O\left(\frac{1}{n}\right),\end{aligned}$$

where the first equality is due to Lemma 1, and

$$\begin{aligned}\Pr\left(\bigcap_{j=1}^{n}\{J(j) \leq (1+\epsilon_1')E[J]\}\right) \geq& \prod_{j=1}^{n}\Pr(J(j) \leq (1+\epsilon_1')E[J]) \\ =&\left(1-O\left(\frac{1}{n^2}\right)\right)^n \\ =&1-O\left(\frac{1}{n}\right).\end{aligned}$$

This implies that C_{ij} is stochastically lower bounded by C_{ij}' and stochastically upper bounded by C_{ij}'' with probability $1-O(\frac{1}{n})$. Hence, in order to show (18) and (19), it suffices to show

$$\Pr(C_k' \leq (1-\epsilon)E[C_k']) \leq \exp\left\{-\frac{E[C_k']\epsilon^2}{2}\right\} \qquad (20)$$

and

$$\Pr(C_k'' \geq (1+\epsilon)E[C_k'']) \leq \exp\left\{-\frac{E[C_k'']\epsilon^2}{3}\right\}. \qquad (21)$$

In $G'(\mathcal{X}, P_0, \gamma)$ and $G''(\mathcal{X}, P_0, \gamma)$, the SINR of link (i, j) is given by (16) and (17), respectively, and because d_{ij}'s for a given i are independent, by applying the Chernoff bounds, we obtain (20) and (21). □

Since C_{ij}' and C_{ij}'' are asymptotically equal to C_{ij}, $E[C_k']$ and $E[C_k'']$ are asymptotically equal to $E[C_k]$. Consequently, Lemma 2 shows that C_k concentrates at $E[C_k]$ asymptotically almost surely (a.a.s.).

Now, let $C_{s,t}$ be the minimum cut capacity among all s-t-cuts, i.e.,

$$C_{s,t} = \min_{0\leq k\leq m} C_k. \qquad (22)$$

[1] In context of graph theory, an event A is called increasing if $I_A(G) \leq I_A(G')$ whenever graph G is a subgraph of G', where I_A is the indicator function of A. An event A is called decreasing if A^c is increasing. For details, please see [2], [13], [15].

For the given source node s and the sets of destination nodes $\mathcal{T} = \{t_1, ..., t_l\}$ and relay nodes $\mathcal{R} = \{u_1, ..., u_m\}$, define the network coding capacity as

$$C_{s,\mathcal{T}} = \min_{t \in \mathcal{T}} C_{s,t}. \tag{23}$$

That is because for one source and multiple destinations, the capacity of network coding depends on the minimum cut among all the destinations.

In the following, we show that when the number of relay nodes m is sufficiently large, the network coding capacity $C_{s,\mathcal{T}}$ concentrates at $E[C_0] = m\bar{C}$ with high probability.

Theorem 3: When n is sufficiently large, with high probability, the network coding capacity $C_{s,\mathcal{T}}$ satisfies

$$\Pr(C_{s,\mathcal{T}} \geq (1-\epsilon'_\alpha)E[C_0]) = 1 - O\left(\frac{l}{m^\alpha}\right), \tag{24}$$

where $\epsilon'_\alpha = \sqrt{\frac{2\alpha \ln m}{E[C_0]}}$ for $\alpha > 0$ and $E[C_0] = m\bar{C}$.

Proof: Since the C_{ij}'s are asymptotically equal to C'_{ij}'s, in order to show (24), it is equivalent to show

$$\Pr(C_{s,\mathcal{T}} \geq (1-\epsilon'_\alpha)E[C'_0]) = 1 - O\left(\frac{l}{m^\alpha}\right).$$

Since $E[C'_k] \geq E[C'_0]$ for any $k = 1, ..., m$,

$$\Pr(C_{s,t} \leq (1-\epsilon'_\alpha)E[C'_0]) \leq \Pr(C_{s,t} \leq (1-\epsilon'_\alpha)E[C'_{k'}]),$$

for any $t \in \mathcal{T}$, where k' is the size of the minimum s-t-cut. By (18) of Lemma 2, we have

$$\begin{aligned}\Pr(C_{s,t} \leq (1-\epsilon'_\alpha)E[C'_{k'}]) &\leq \exp\left\{-\frac{\epsilon'^2_\alpha[m+k'(m-k')]\bar{C}'}{2}\right\} \\ &\leq \exp\left\{-\frac{\epsilon'^2_\alpha m\bar{C}'}{2}\right\}.\end{aligned}$$

By choosing $\epsilon'_\alpha = \sqrt{\frac{2\alpha \ln m}{E[C_0]}}$, since $\bar{C}'$ and $\bar{C}$ are asymptotically equal, we have for any $t \in \mathcal{T}$,

$$\Pr(C_{s,t} \leq (1-\epsilon'_\alpha)E[C'_0]) = O\left(\frac{1}{m^\alpha}\right).$$

By the union bound, we have

$$\begin{aligned}\Pr(C_{s,\mathcal{T}} \leq (1-\epsilon'_\alpha)E[C'_0]) &\leq \sum_{t\in\mathcal{T}} \Pr(C_{s,t} \leq (1-\epsilon'_\alpha)E[C'_0]) \\ &= O\left(\frac{l}{m^\alpha}\right).\end{aligned}$$

□

Theorem 4: When n is sufficiently large, with high probability, the network coding capacity $C_{s,\mathcal{T}}$ satisfies

$$\Pr(C_{s,\mathcal{T}} \leq (1+\epsilon''_\alpha)E[C_0]) = 1 - O\left(\frac{1}{m^\alpha}\right), \tag{25}$$

where $\epsilon''_\alpha = \sqrt{\frac{3\alpha \ln m}{E[C_0]}}$ for $\alpha > 0$ and $E[C_0] = m\bar{C}$.

Proof: Since the C_{ij}'s are asymptotically equal to C''_{ij}'s, in order to show (25), it is equivalent to show

$$\Pr(C_{s,\mathcal{T}} \leq (1+\epsilon''_\alpha)E[C''_0]) = 1 - O\left(\frac{1}{m^\alpha}\right).$$

To show this, it is sufficient to consider a particular cut for a pair of the source and one destination, e.g., an s-t-cut separating the source s from all the other nodes.

$$\begin{aligned}\Pr(C_{s,\mathcal{T}} \geq (1+\epsilon''_\alpha)E[C''_0]) &\leq \Pr\left(C_{s,t} \geq (1+\epsilon''_\alpha)E[C''_0]\right) \\ &\leq \Pr\left(\sum_{i=1}^{m} C_{si} \geq (1+\epsilon''_\alpha)E[C''_0]\right) \\ &= \Pr(C_0 \geq (1+\epsilon''_\alpha)E[C''_0]) \\ &\leq \exp\left(-\frac{\epsilon''^2_\alpha m\bar{C}''}{3}\right) \\ &= O\left(\frac{1}{m^\alpha}\right).\end{aligned}$$

where the last inequality follows from (19) of Lemma 2. □

C. Heterogeneous Transmission Powers

In this subsection, we consider the case where the transmission power of each node is random rather than a constant, but the capacity of a link (i,j) is a constant R, which is independent of the SINR β_{ij}, when $\beta_{ij} \geq \beta$. In this case, β_{ij} can be rewritten as

$$\begin{aligned}\beta_{ij} &= \frac{P_i L(d_{ij})}{N_0 + \gamma \sum_{k\neq i,j} P_k L(d_{kj})} \\ &= \frac{P_i L(d_{ij})}{N_0 + \gamma I(j) - \gamma P_i L(d_{ij})}.\end{aligned} \tag{26}$$

Because P_i's and $\mathbf{X}_i$'s are both i.i.d., using the same method, we can prove the following lemma:

Lemma 5: Suppose there are n nodes in the network, then

$$\Pr(I(j) \leq (1-\epsilon_2)E[I]) = O\left(\frac{1}{n^2}\right), \tag{27}$$

and

$$\Pr(I(j) \geq (1+\epsilon'_2)E[I]) = O\left(\frac{1}{n^2}\right), \tag{28}$$

for all $j = 1, 2, ..., n$, where $\epsilon_2 = \sqrt{\frac{4\ln n}{(n-1)E[P]E[L]}}$ and $\epsilon'_2 = \sqrt{\frac{6\ln n}{(n-1)E[P]E[L]}}$.

Even though we have concentration results for $I(j)$, we cannot employ the same coupling methods as in the previous section. This is because in $G'(\mathcal{X}, P_0, \gamma)$ (or $G''(\mathcal{X}, P_0, \gamma)$), the C'_{ij}'s (respectively, C''_{ij}'s) are independent for all $j \neq i$ for given i. In our new case, however, this independence does not hold because all C_{ij}'s depend on transmission power P_i. To deal with this dependence, we use martingale methods and Azuma's inequality to solve our problem.

Theorem 6 (Azuma's Inequality [2]): Let $Z_0, Z_1, ...,$ be a martingale sequence such that for each $i = 1, 2, ...,$,

$$|Z_i - Z_{i-1}| \leq c_i$$

almost surely, where c_i may depend on i. Then for all $n > 0$ and any $\lambda > 0$,

$$\Pr(Z_n \geq Z_0 + \lambda) \leq \exp\left\{-\frac{\lambda^2}{2\sum_{i=1}^n c_i^2}\right\}, \quad (29)$$

and

$$\Pr(Z_n \leq Z_0 - \lambda) \leq \exp\left\{-\frac{\lambda^2}{2\sum_{i=1}^n c_i^2}\right\}. \quad (30)$$

Proof: Please see e.g. [2]. □

To use Azuma's inequality, we need to construct a martingale. A common approach to obtain a martingale from a sequence of random variables (not necessarily independent) is to construct a Doob sequence. More precisely, suppose we have a sequence of random variables $Y_1, Y_2, ..., Y_n$, which are not necessarily independent. Let $S = \sum_i^n Y_i$ and define a new sequence of random variables $\{Z_i : i = 0, 1, ..., n\}$ by:

$$\begin{cases} Z_0 = E[S] \\ Z_i = E_{Y_{i+1},...,Y_n}[S|Y_1, ..., Y_i], \quad i = 1, 2, ..., n. \end{cases} \quad (31)$$

Then $\{Z_i : i = 0, 1, ..., n\}$ is a martingale and $Z_n = S$.

If we are able to upper bound the difference $|Z_i - Z_{i-1}|$ for all i by some constant, then we can apply Azuma's inequality to obtain some bound on a tail probability. For example, if Y_i's are independent, a simple upper bound for $|Z_i - Z_{i-1}|$ is any upper bound on $|Y_i|$. However, as long as the Y_i's are dependent, which is the case in our model, we cannot bound $|Z_i - Z_{i-1}|$ in this way. In this case, we need to understand the properties of the Y_i's to see if we can bound $|Z_i - Z_{i-1}|$. We approach our problem by following this idea and using the next corollary.

Lemma 7: For $n > 1$, given a sequence of random variables $Y_1, Y_2, ..., Y_n$, which are not necessarily independent, let $S = \sum_i^n Y_i$. If for any $y_i, y_i' \in D_i$, where D_i is the support of Y_i,

$$|E[S|Y_1, ..., Y_{i-1}, Y_i = y_i] - E[S|Y_1, ..., Y_{i-1}, Y_i = y_i']| \leq c_i,$$

almost surely, where c_i may depend on i, then for any $\lambda > 0$,

$$\Pr(S \geq E[S] + \lambda) \leq \exp\left\{-\frac{\lambda^2}{2\sum_{i=1}^n c_i^2}\right\}, \quad (32)$$

and

$$\Pr(S \leq E[S] - \lambda) \leq \exp\left\{-\frac{\lambda^2}{2\sum_{i=1}^n c_i^2}\right\}. \quad (33)$$

Proof: We prove this corollary for the case of discrete random variables. For continuous random variables, the proof is similar.

Define a Doob sequence with respect to $\{Y_i : i = 1, ..., n\}$ as in (31). To simplify the notation, we will write $E_{Y_{i+1},...,Y_n}[S|Y_1, ..., Y_i]$ as $E[S|Y_1, ..., Y_i]$ when there is ambiguity.

By the total conditional probability theorem, we have

$$\begin{aligned} Z_{i-1} &= E[S|Y_1, ..., Y_{i-1}] \\ &= \sum_{y \in D_i} E[S|Y_1, ..., Y_{i-1}, Y_i = y] \Pr(Y_i = y|Y_1, ..., Y_{i-1}), \end{aligned}$$

and

$$\begin{aligned} Z_i &= E[S|Y_1, ..., Y_i] \\ &= \sum_{y \in D_i} E[S|Y_1, ..., Y_i] \Pr(Y_i = y|Y_1, ..., Y_{i-1}). \end{aligned}$$

Therefore,

$$\begin{aligned} & |Z_i - Z_{i-1}| \\ = & |E[S|Y_1, ..., Y_i] - E[S|Y_1, ..., Y_{i-1}]| \\ = & |\sum_{y \in D_i} E[S|Y_1, ..., Y_i] \Pr(Y_i = y|Y_1, ..., Y_{i-1}) \\ & - \sum_{y \in D_i} E[S|Y_1, ..., Y_{i-1}, Y_i = y] \Pr(Y_i = y|Y_1, ..., Y_{i-1}) \\ \leq & \sum_{y \in D_i} |E[S|Y_1, ..., Y_i] - E[S|Y_1, ..., Y_{i-1}, Y_i = y]| \\ & \cdot \Pr(Y_i = y|Y_1, ..., Y_{i-1}) \\ \leq & \sum_{y \in D_i} c_i \Pr(Y_i = y|Y_1, ..., Y_{i-1}) \\ = & c_i. \end{aligned}$$

Since $\{Z_i : i = 0, 1, ..., n\}$ is a martingale with bounded difference of $|Z_i - Z_{i-1}|$, we can apply Azuma's inequality to obtain (32) and (33). □

Now consider $G'(\mathcal{X}, \mathcal{P}, \gamma)$ and $G''(\mathcal{X}, \mathcal{P}, \gamma)$ coupled with $G(\mathcal{X}, \mathcal{P}, \gamma)$ such that they have the same point process $\mathcal{X}$ and powers $\mathcal{P}$. Then, the SINR of link (i, j) in $G'(\mathcal{X}, \mathcal{P}, \gamma)$ and $G''(\mathcal{X}, \mathcal{P}, \gamma)$ are

$$\beta_{ij}' = \frac{P_i L(d_{ij})}{N_0 + (1 + \epsilon_2')\gamma E[I] - \gamma P_i L(d_{ij})} \quad (34)$$

and

$$\beta_{ij}'' = \frac{P_i L(d_{ij})}{N_0 + (1 - \epsilon_2)\gamma E[I] - \gamma P_i L(d_{ij})}, \quad (35)$$

respectively.

Let C_{ij}' and C_{ij}'' be the capacity of link (i, j) in $G'(\mathcal{X}, \mathcal{P}, \gamma)$ and $G''(\mathcal{X}, \mathcal{P}, \gamma)$, respectively. Then, C_{ij}' and C_{ij}'' are asymptotically equal to C_{ij}.

Assume that there exist $r_{min}' > 0$, $r_{max}' > 0$, $r_{min}'' > 0$ and $r_{max}'' > 0$ as the solutions for

$$\frac{p_{min} L(r_{min}')}{N_0 + \gamma(1 + \epsilon_2')E[I] - \gamma p_{min} L(r_{min}')} = \beta,$$

$$\frac{p_{max} L(r_{max}')}{N_0 + \gamma(1 + \epsilon_2')E[I] - \gamma p_{max} L(r_{max}')} = \beta,$$

$$\frac{p_{min} L(r_{min}'')}{N_0 + \gamma(1 - \epsilon_2)E[I] - \gamma p_{min} L(r_{min}'')} = \beta,$$

and

$$\frac{p_{max} L(r_{max}'')}{N_0 + \gamma(1 - \epsilon_2)E[I] - \gamma p_{max} L(r_{max}'')} = \beta,$$

respectively. That is

$$r_{min}' = L^{-1}\left(\frac{\beta}{1 + \gamma\beta} \cdot \frac{N_0 + \gamma(1 + \epsilon_2')E[I]}{p_{min}}\right),$$

$$r'_{max} = L^{-1}\left(\frac{\beta}{1+\gamma\beta}\cdot\frac{N_0+\gamma(1+\epsilon'_2)E[I]}{p_{max}}\right),$$
$$r''_{min} = L^{-1}\left(\frac{\beta}{1+\gamma\beta}\cdot\frac{N_0+\gamma(1-\epsilon_2)E[I]}{p_{min}}\right),$$
$$r''_{max} = L^{-1}\left(\frac{\beta}{1+\gamma\beta}\cdot\frac{N_0+\gamma(1-\epsilon_2)E[I]}{p_{max}}\right).$$

Since $L(\cdot)$ is continuous and strictly decreasing, r'_{min}, r'_{max}, r''_{min} and r''_{max} are all unique. In $G'(\mathcal{X},\mathcal{P},\gamma)$ ($G''(\mathcal{X},\mathcal{P},\gamma)$), any node inside the circle centered at $\mathbf{X}_i$ with radius r'_{min} (r''_{min}) is connected to node i by a bidirectional link; while any node outside the circle centered at $\mathbf{X}_i$ with radius r'_{max} (r''_{max}) is not connected to node i.

Let $\mathcal{A}(\mathbf{X}_i, r'_{min}, r'_{max})$ and $\mathcal{A}(\mathbf{X}_i, r''_{min}, r''_{max})$ be the two annuli with inner radius r'_{min} and outer radius r'_{max}, and inner radius r''_{min} and outer radius r''_{max}, respectively. Denote by $N(r'_{min}, r'_{max})$ and $N(r''_{min}, r''_{max})$ the number of nodes in $\mathcal{A}(\mathbf{X}_i, r'_{min}, r'_{max})$ and $\mathcal{A}(\mathbf{X}_i, r''_{min}, r''_{max})$, respectively. It is clear that $N(r'_{min}, r'_{max})$ and $N(r''_{min}, r''_{max})$ have Poisson distribution with mean $n\pi(r'^2_{max}-r'^2_{min})$ and $n\pi(r''^2_{max}-r''^2_{min})$, respectively.

Now suppose the signal attenuation function $L(x)=cx^{-\alpha}$ for some constants $c>0$ and $2<\alpha<4$. Then,

$$r'_{min} = \left(\frac{c(1+\gamma\beta)p_{min}}{\beta[N_0+\gamma(1+\epsilon'_2)E[I]]}\right)^{\alpha},$$
$$r'_{max} = \left(\frac{c(1+\gamma\beta)p_{max}}{\beta[N_0+\gamma(1+\epsilon'_2)E[I]]}\right)^{\alpha},$$
$$r''_{min} = \left(\frac{c(1+\gamma\beta)p_{min}}{\beta[N_0+\gamma(1-\epsilon_2)E[I]]}\right)^{\alpha},$$
$$r''_{max} = \left(\frac{c(1+\gamma\beta)p_{max}}{\beta[N_0+\gamma(1-\epsilon_2)E[I]]}\right)^{\alpha},$$

and

$$r'^2_{max}-r'^2_{min} = \frac{B(p_{min},p_{max})}{[N_0+\gamma(1+\epsilon'_2)E[I]]^{2\alpha}}, \quad (36)$$
$$r''^2_{max}-r''^2_{min} = \frac{B(p_{min},p_{max})}{[N_0+\gamma(1-\epsilon_2)E[I]]^{2\alpha}}, \quad (37)$$

where

$$B(p_{min},p_{max}) = (p^{2\alpha}_{max}-p^{2\alpha}_{min})\left[\frac{c(1+\gamma\beta)}{\beta}\right]^{2\alpha}. \quad (38)$$

¿From (36) and (37), we can see that both $n\pi(r'^2_{max}-r'^2_{min})$ and $n\pi(r''^2_{max}-r''^2_{min})$ scale with n as $\frac{B}{n^{2\alpha-1}}$, since $E[I]$ scales linear with n. Now assume that there exists a constant $\eta>0$ independent of n such that

$$N(r'_{min}, r'_{max}) \le \eta, \quad \text{and} \quad N(r''_{min}, r''_{max}) \le \eta \quad (39)$$

hold a.a.s. This assumption actually puts a constraint on the transmission power since it needs to scale (if it scales) with n so that (39) is satisfied. For example, we may choose $\eta=1$ and the transmission power P scales with n so that $B(p_{min},p_{max})=\Theta(\frac{n^{2\alpha-1}}{\log n})$. Note that r'_{min} and r'_{max} are asymptotically equal to r''_{min} and r''_{max}, respectively.

The following lemma establishes a concentration result for C_k with heterogeneous transmission power and constant capacity by coupling methods and Azuma's inequality.

Lemma 8: For any $0<\epsilon<1$, when n is sufficiently large and (39) is guaranteed, with high probability, the capacity of an s-t-cut of size $k, k=0,1,...,m$, satisfies

$$\Pr(C_k \le (1-\epsilon)E[C'_k]) \le \exp\left\{-\frac{[m+k(m-k)]\bar{C}'^2\epsilon^2}{2(\eta+1)^2R^2}\right\}, \quad (40)$$

where $E[C'_k]=[m+k(m-k)]\bar{C}'$ and $\bar{C}'$ is the average link capacity in $G'(\mathcal{X},\mathcal{P},\gamma)$, and

$$\Pr(C_k \ge (1+\epsilon)E[C''_k]) \le \exp\left\{-\frac{[m+k(m-k)]\bar{C}''^2\epsilon^2}{2(\eta+1)^2R^2}\right\}, \quad (41)$$

where $E[C''_k]=[m+k(m-k)]\bar{C}''$ and $\bar{C}''$ is the average link capacity in $G''(\mathcal{X},\mathcal{P},\gamma)$.

Proof: By Lemma 5, for all j, $(1-\epsilon_2)E[I] \le I(j)) \le (1+\epsilon'_2)E[I]$ holds a.a.s. It is clear that C_{ij} is stochastically lower bounded by C'_{ij} and stochastically upper bounded by C''_{ij} almost surely. Hence, in order to show (40) and (41), it suffices to show

$$\Pr(C'_k \le (1-\epsilon)E[C'_k]) \le \exp\left\{-\frac{[m+k(m-k)]\bar{C}'^2\epsilon^2}{2(\eta+1)^2R^2}\right\} \quad (42)$$

and

$$\Pr(C''_k \ge (1+\epsilon)E[C''_k]) \le \exp\left\{-\frac{[m+k(m-k)]\bar{C}''^2\epsilon^2}{2(\eta+1)^2R^2}\right\}. \quad (43)$$

To show (42), we use martingale methods. Let $Y_1=C'_{s(k+1)}, Y_2=C'_{s(k+2)},...,Y_{m-k}=C'_{sm}$, and $Y_{m-k+1}=C'_{1(k+1)}, Y_{m-k+2}=C'_{1(k+2)},...,Y_{m-k+k(m-k)}=C'_{km}$, and $Y_{m-k+k(m-k)+1}=C'_{1t}, Y_{m-k+k(m-k)+2}=C'_{2t},...,Y_{m-k+k(m-k)+k}=C'_{kt}$. Define a Doob sequence $\{Z_i : i=0,...,m+k(m-k)\}$ with respect to $\{Y_i, i=1,2,...,m+k(m-k)\}$ as

$$\begin{cases} Z_0 = E[C'_k] \\ Z_i = E[C'_k|Y_1,...,Y_i], \quad i=1,2,...,m+k(m-k). \end{cases}$$

Then $\{Z_i : i=0,...,m+k(m-k)\}$ is a martingale and $Z_{m+k(m-k)}=C'_k$.

Since when $i\ne u$ and $j\ne v$, C'_{ij} is independent of C'_{uv}, we have only the dependence among C'_{ij}'s for all $j\ne i$ with given i. However, the distance d_{ij}'s are independent for all $j\ne i$ with given i. When $d_{ij}\le r'_{min}$, $C'_{ij}=R$, and when $d_{ij}>r'_{max}$, $C'_{ij}=0$. Moreover, the number of nodes within the annulus $\mathcal{A}(\mathbf{X}_i, r'_{min}, r'_{max})$ is upper bounded by the constant η a.a.s. Therefore, we have

$$|E[C'_k|Y_1,...,Y_{i-1},Y_i=y_i]-E[C'_k|Y_1,...,Y_{i-1},Y_i=y'_i]| \le (\eta+1)R$$

a.a.s., where y_i and y'_i are either 0 or R. Applying the result of Lemma 7, we have (42). In the same manner, we can show that (43) holds. □

In the following, we show that as the number of relay nodes m is sufficiently large, the network coding capacity $C_{s,\mathcal{T}}$ concentrates at $E[C_0]=m\bar{C}$ with high probability. The proofs are similar to those for Theorem 3 and Theorem 4.

Theorem 9: When n is sufficiently large, with high probability, the network coding capacity $C_{s,\mathcal{T}}$ satisfies

$$\Pr(C_{s,\mathcal{T}} \geq (1-\epsilon_\alpha)E[C_0]) = 1 - O\left(\frac{l}{m^\alpha}\right), \quad (44)$$

where $\epsilon_\alpha = \frac{(\eta+1)R}{E[C_0]}\sqrt{2\alpha m \ln m}$ for $\alpha > 0$ and $E[C_0] = m\bar{C}$.

Proof: Since C_{ij}'s are asymptotically equal to C'_{ij}'s, in order to show (44), it is equivalent to show

$$\Pr(C_{s,\mathcal{T}} \geq (1-\epsilon_\alpha)E[C'_0]) = 1 - O\left(\frac{l}{m^\alpha}\right).$$

Since $E[C'_k] \geq E[C'_0]$ for any $k = 1, ..., m$,

$$\Pr(C_{s,t} \leq (1-\epsilon_\alpha)E[C'_0]) \leq \Pr(C_{s,t} \leq (1-\epsilon_\alpha)E[C'_{k'}]),$$

for any $t \in \mathcal{T}$, where k' is the size of the minimum s-t-cut. By (40) of Lemma 8, we have

$$\begin{aligned}\Pr(C_{s,t} \leq (1-\epsilon_\alpha)E[C'_{k'}]) &\leq \exp\left\{-\frac{\epsilon_\alpha^2[m+k'(m-k')]\bar{C}'^2}{2(\eta+1)^2R^2}\right\} \\ &\leq \exp\left\{-\frac{\epsilon_\alpha^2 m\bar{C}'^2}{2(\eta+1)^2R^2}\right\}.\end{aligned}$$

By choosing $\epsilon_\alpha = \frac{(\eta+1)R}{E[C_0]}\sqrt{2\alpha m \ln m}$ for $\alpha > 0$, since $\bar{C}'$ and $\bar{C}$ are asymptotically equal, for any $t \in \mathcal{T}$,

$$\Pr(C_{s,t} \leq (1-\epsilon_\alpha)E[C'_0]) = O\left(\frac{1}{m^\alpha}\right).$$

By the union bound, we have

$$\begin{aligned}\Pr(C_{s,\mathcal{T}} \leq (1-\epsilon_\alpha)E[C'_0]) &\leq \sum_{t\in\mathcal{T}} \Pr(C_{s,t} \leq (1-\epsilon_\alpha)E[C'_0]) \\ &= O\left(\frac{l}{m^\alpha}\right).\end{aligned}$$

□

Theorem 10: When n is sufficiently large, with high probability, the network coding capacity $C_{s,\mathcal{T}}$ satisfies

$$\Pr(C_{s,\mathcal{T}} \leq (1+\epsilon_\alpha)E[C_0]) = 1 - O\left(\frac{1}{m^\alpha}\right), \quad (45)$$

where $\epsilon_\alpha = \frac{(\eta+1)R}{E[C_0]}\sqrt{2\alpha m \ln m}$ for $\alpha > 0$ and $E[C_0] = m\bar{C}$.

Proof: Since C_{ij}'s are asymptotically equal to C''_{ij}'s, in order to show (45), it is equivalent to show

$$\Pr(C_{s,\mathcal{T}} \leq (1+\epsilon_\alpha)E[C''_0]) = 1 - O\left(\frac{1}{m^\alpha}\right).$$

To show this, it is sufficient to consider a particular cut for a pair of the source and one destination, for instance, an s-t-cut separating the source s from all the other nodes.

$$\begin{aligned}\Pr(C_{s,\mathcal{T}} \geq (1+\epsilon_\alpha)E[C''_0]) &\leq \Pr\left(C_{s,t} \geq (1+\epsilon_\alpha)E[C''_0]\right) \\ &\leq \Pr\left(\sum_{i=1}^m C_{si} \geq (1+\epsilon_\alpha)E[C''_0]\right) \\ &= \Pr(C_0 \geq (1+\epsilon_\alpha)E[C''_0]) \\ &\leq \exp\left\{-\frac{\epsilon_\alpha^2 m\bar{C}''^2}{2(\eta+1)^2R^2}\right\} \\ &= O\left(\frac{1}{m^\alpha}\right).\end{aligned}$$

where the last inequality follows from (41) of Lemma 8. □

IV. Network Coding Capacity for Multiple-Source Transmission

In this section, we study network coding capacity for multiple sources and multiple destinations transmission. We assume the same notation as in Section III. However, instead of having a single source, we have $s \geq 2$ sources. Denote by $\mathcal{S} = \{s_1, \ldots, s_h\}$ the set of source nodes. Assume there is no correlation among the set of sources $\mathcal{S}$. Now we can define an $\mathcal{S}$-t-cut of size k between the set of sources $\mathcal{S}$ and one destination $t \in \mathcal{T}$ as a partition of the relay nodes into two sets V_k and V_k^c, such that $|V_k| = k, |V_k^c| = m-k$, $V_k \cup V_k^c = \mathcal{R}$ and $V_k \cap V_k^c = \emptyset$. Let

$$C_k = \sum_{i=1}^h \sum_{u_j\in V_k^c} C_{s_i j} + \sum_{u_j\in V_k}\sum_{u_i\in V_k^c} C_{ji} + \sum_{u_j\in V_k} C_{jt}, \quad (46)$$

then C_k is the capacity of the corresponding $\mathcal{S}$-t-cut, and

$$\begin{aligned}E[C_k] &= E_X E_P[C_k] \\ &= [(m-k)h + k(m-k) + k]\bar{C}, \quad (47)\end{aligned}$$

Now, let $C_{\mathcal{S},t}$ be the minimum cut capacity among all $\mathcal{S}$-t-cuts,

$$C_{\mathcal{S},t} = \min_{0\leq k\leq m} C_k. \quad (48)$$

By comparing (6) and (47), we note that we on longer have symmetry in $E[C_k]$ with respect to k, i.e., $E[C_k] \neq E[C_{m-k}]$ for $k = 0, 1, ..., m$. In the single source case, the minimum value of $E[C_k]$, i.e., $E[C_{s,t}]$ is obtained when $k = 0$ or $k = m$ due to the symmetry ($E[C_0] = E[C_m]$). This means that the bottlenecks are at the source end and also the destination end. Nevertheless, when we have multiple sources, $E[C_0] > E[C_m]$, and the minimum expectation value of the capacity among all cuts with any size is $E[C_{\mathcal{S},t}] = E[C_m]$, which implies that we have only one bottleneck at the destination end.

For the given set of source nodes $\mathcal{S} = \{s_1, ..., s_s\}$ and the sets of destination nodes $\mathcal{T} = \{t_1, ..., t_l\}$ and relay nodes $\mathcal{R} = \{u_1, ..., u_m\}$, define the network coding capacity for multiple sources and multiple destinations as

$$C_{\mathcal{S},\mathcal{T}} = \min_{t\in\mathcal{T}} C_{\mathcal{S},t}. \quad (49)$$

Then, by the same method used in the previous section, we can show that $C_{\mathcal{S},\mathcal{T}}$ concentrates at $E[C_m] = m\bar{C}$ with high probability, where $\bar{C}$ is defined the same as before. This indicates that $C_{\mathcal{S},\mathcal{T}}$ and $C_{s,\mathcal{T}}$ concentrate at the same value. This is because they have one bottleneck in common.

Theorem 11: When n is sufficiently large, the network coding capacity $C_{\mathcal{S},\mathcal{T}}$ satisfies

$$\Pr(C_{\mathcal{S},\mathcal{T}} \geq (1-\epsilon_\alpha)E[C_m]) = 1 - O\left(\frac{l}{m^\alpha}\right), \quad (50)$$

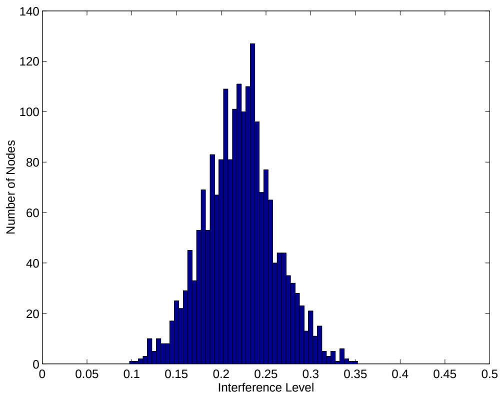

Fig. 3. Interference at each node in $G(\mathcal{X}, P_0, \gamma)$

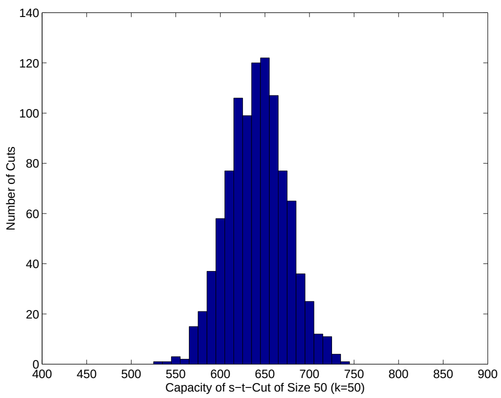

Fig. 4. Capacity of random s-t-cut of size $k = 50$ in $G(\mathcal{X}, P_0, \gamma)$

where $\epsilon_\alpha = \frac{(\eta+1)R}{E[C_m]}\sqrt{2\alpha m \ln m}$ for $\alpha > 0$ and $E[C_m] = m\bar{C}$.

Proof: The proof is the same as that for Theorem 9 by replacing of $E[C_0]$ by $E[C_m]$ □

Theorem 12: When n is sufficiently large, the network coding capacity $C_{\mathcal{S},\mathcal{T}}$ satisfies

$$\Pr(C_{\mathcal{S},\mathcal{T}} \leq (1+\epsilon_\alpha)E[C_m]) = 1 - O\left(\frac{1}{m^\alpha}\right), \qquad (51)$$

where $\epsilon_\alpha = \frac{(\eta+1)R}{E[C_m]}\sqrt{2\alpha m \ln m}$ for $\alpha > 0$ and $E[C_m] = m\bar{C}$.

Proof: The proof is the same as that for Theorem 10 by replacing of $E[C_0]$ by $E[C_m]$ □

V. Simulation Studies

In this section, we present some simulation results on the SINR model and network coding capacity. Fig. 3 and Fig. 4 show simulation results on interference and cut capacity in $G(\mathcal{X}, P_0, \gamma)$, where $n = 2000$, $L(x) = \frac{10^{-3}}{64}x^{-3}$, $N_0 = 0.02$, $\beta = 0.2$ and $\gamma = 0.02$, and every node transmits with constant power $P_0 = 0.01$. Fig. 5 and Fig. 6 show simulation results on interference and cut capacity in $G(\mathcal{X}, \mathcal{P}, \gamma)$, where

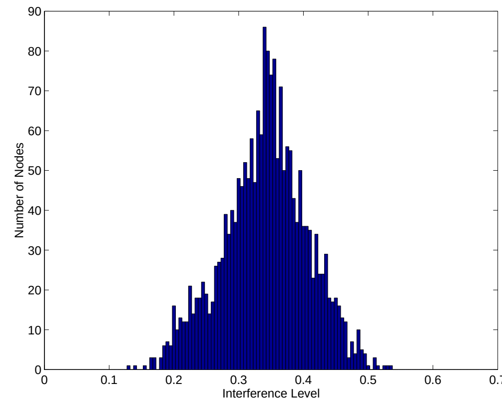

Fig. 5. Interference at each node in $G(\mathcal{X}, \mathcal{P}, \gamma)$

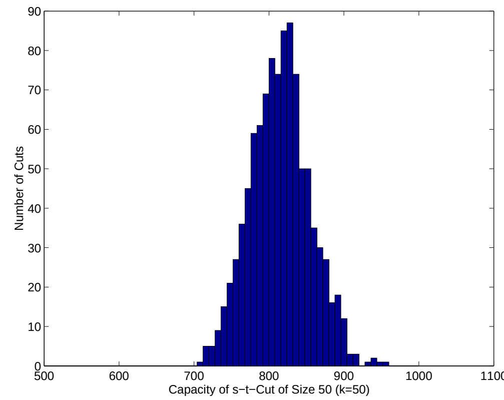

Fig. 6. Capacity of random s-t-cut of size $k = 50$ in $G(\mathcal{X}, \mathcal{P}, \gamma)$

$n = 2000$, $L(x) = \frac{10^{-3}}{64}x^{-3}$, $N_0 = 0.02$, $\beta = 0.2$ and $\gamma = 0.02$, and every node transmits with power P uniformly randomly distributed over $[0.01, 0.02]$. The results confirm the concentration behavior of interference and cut capacity.

VI. Conclusions

In this paper, we studied network coding capacity for random wireless networks with interference and noise. In this model, the capacities of links are not independent. By using coupling and martingale methods, we showed that when the size of the network is sufficiently large, the network coding capacity still exhibits a concentration behavior in cases of single source multiple destinations and multiple sources multiple destinations. We demonstrated simulation results that meet our theoretical bounds of network coding capacity.

References

[1] A. Ahlswede, N. Cai, S.Y.R. Li, and R. Y. Yeung. Network information flow. *IEEE Trans. on Information Theory*, 46(4):1204–1216, July 2000.

[2] N. Alon and J. Spencer. *The Proabbilistic Methods*. John Wiley, New York, second edition, 2000.

[3] S. A. Aly, V. Kapoor, J. Meng, and A. Klappenecker. Bounds on the network coding capacity for wireless random networks. In *Proc. 3rd Workshop on Network Coding, Theory, and Applications*, San Diego, CA, U.S.A., 2007.

[4] B. Bollobás. *Random Graphs*. Academic Press, New York, second edition, 2001.

[5] T. M. Cover and J. A. Thomas. *Elements of Information Theory*. Wiley-Interscience, New York, 1991.

[6] J. Diaz, M. D. Penrose, J. Petit, and M. Serna. Approximating layout problems on random geometric graphs. *Journal of Algorithms*, 39(1):78–116, 2001.

[7] O. Dousse, M. Franceschetti, and P. Thiran. Information theoretic bounds on the throughput scaling of wireless relay networks. In *Proc. of the IEEE INFOCOM'05*, Mar. 2005.

[8] T. Ho, R. Koetter, M. Médard, M. Effros, J. Shi, and D. Karger. A random linear network coding approach to multicast. *IEEE Trans. on Information Theory*, 52(10):4413–4430, Oct. 2006.

[9] S. Jaggi, P. Sanders, P. A. Chou, M. Effros, S. Enger, and L. Tolhuizen. Polynomial time algorithm s for multicast network code construction. *IEEE Trans. on Information Theory*, 51(6):1973–1982, June 2005.

[10] S. Janson, T. Luczak, and A. Ruciński. *Random Graphs*. John Wiley & Sons, New York, 2000.

[11] R. Koetter and M. Médard. Algebraic approach to network coding. *IEEE/ACM Trans. on Networking*, 11(5):782–795, Oct. 2003.

[12] S. Y. R. Li, R. W. Yeung, and N. Cai. Linear network coding. *IEEE Trans. on Information Theory*, 49(2):371–381, Feb. 2003.

[13] R. Meester and R. Roy. *Continuum Percolation*. Cambridge University Press, New York, 1996.

[14] R. Motwani and P. Raghavan. *Randomized Algorithms*. Cambridge University Press, 1995.

[15] M. Penrose. *Random Geometric Graphs*. Oxford University Press, New York, 2003.

[16] J. Proakis. *Digital Communications*. MaGray Hill, 4th edition, 2000.

[17] A. Ramamoorthy, J. Shi, and R. D. Wesel. On the capacity of network coding for random networks. *IEEE Trans. on Information Theory*, 51(8):2878–2885, Aug. 2005.

[18] A. Ramamoorthy, J. Shi, and R. D. Wesel. On the capacity of network coding for random networks. In *Proc. 41st Allerton Conference on Communication, Control and Computing*, Monticello, IL, U.S.A., 2003.

[19] D. Tse and P. Viswanath. *Fundamentals of Wireless Communication*. Cambridge University Press, 2005.